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A. SCHRIJVER

GRAPHS AND SUPERCOMPACT SPACES

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GRAPHS AND SUPERCOMPACT SPACES

by

A. Schrijver.

ABSTRACT.

Some relations between supercompact T_1-spaces and undirected graphs are investigated.

KEYWORDS & PHRASES: supercompactness, binary subbase, interval space, compact orderable space, graph space, bipartite graph, comparable graph.

0. INTRODUCTION

In this report we elaborate the idea of J. DE GROOT [6] of representing supercompact T_1-spaces by means of graphs.
A topological space is called supercompact iff there exists a so-called binary closed subbase for the topology, i.e. a closed subbase $\mathcal{S}$ such that: if $\mathcal{S}' \subset \mathcal{S}$ and $\bigcap \mathcal{S}' = \emptyset$ then there are S_1 and S_2 in $\mathcal{S}'$ with $S_1 \cap S_2 = \emptyset$.
By ALEXANDER's theorem it is clear that every supercompact space is compact. It is further obvious that every compact orderable space is supercompact and that each product of supercompact spaces is supercompact. Furthermore it can be proven that every interval space of a complete lattice is supercompact and that each compact treelike space is supercompact (see BROUWER & SCHRIJVER [2] or VAN MILL [10]). In [2] also a characterization is given of supercompact spaces by means of so-called interval structures. VERBEEK [12] gives an example of a compact T_1-space which is not supercompact. We do not know whether every compact Hausdorff space is supercompact. DE GROOT [4] conjectured that each compact metric space is supercompact; it appears that this conjecture is still open (the proof in O'CONNOR [11] contains an irreparable error, as was pointed out by J. BRUIJNING -oral communication-).
The relation between supercompact T_1-spaces and undirected graphs which DE GROOT gives in [6] is as follows. Given a supercompact T_1-space X with binary closed subbase $\mathcal{S}$, he constructs a graph with vertex set $\mathcal{S}$ and edge set E, such that:

$$\{S_1,S_2\} \in E \quad \text{iff} \quad S_1 \cap S_2 \neq \emptyset \text{ and } S_1 \neq S_2 \qquad (S_1,S_2 \in \mathcal{S}).$$

Conversely, given a graph G with vertex set V and edge set E, the points of the corresponding space are the maximal complete subsets of V (a subset V' of V is called complete if all $v_1,v_2 \in V'$ satisfy $v_1 = v_2$ or $\{v_1,v_2\} \in E$), and a closed subbase for the topology is the collection $\{B_v \mid v \in V\}$ where for any $v \in V$, B_v is the set of all maximal complete subsets V' of V with $v \in V'$. This subbase is binary and generates a T_1-topology. These constructions define a kind of duality between supercompact T_1-spaces furnished with a preassigned binary closed subbase and a class of graphs, such that, if we

start with a supercompact T_1-space with binary subbase and form the graph corresponding to this space, then the space (and subbase) constructed from this graph is homeomorphic to the original space.
In this report we use a slightly different approach, which appears to have some advantages. To each supercompact T_1-space with binary subbase S we assign a graph which again has vertex set S, but now we take as edge set E the set of all pairs $\{S_1,S_2\}$ with $S_1,S_2 \in S$ and $S_1 \cap S_2 = \emptyset$, i.e. we take in some sense the complementary graph of the graph described above. As a consequence the converse construction (graph to space) must also be modified: the points of the space now are the maximal independent subsets of S ($S' \subset S$ is independent if $\{S_1,S_2\} \notin E$ for $S_1,S_2 \in S'$) in stead of maximal complete subsets.
The advantages of this approach become apparent in the formulation of a number of conditions on graphs ensuring that they generate certain prescribed supercompact spaces. For example, BRUIJNING [3], using a characterization of DE GROOT [5], gives conditions for a graph to generate a product of unit segments I. To this purpose he defines the notion of a comparable graph (see section 1). It turns out that this notion can be characterized more elegantly in terms of the complementary graph. Again, a characterization by DE GROOT & SCHNARE [7] of products of compact orderable spaces becomes clearer in the present approach.

In section 1 we present some preliminary definitions concerning topological spaces and graphs and in section 2 we elaborate the relations between topological spaces and graphs indicated in this introduction.

1. PRELIMINARIES

In this section we give some preliminary definitions and facts concerning topological spaces and undirected graphs.

a. *Topological spaces.*

In this report we always suppose that each subbase S for a topological space X is a *closed* subbase and that $\emptyset \notin S$ and $X \notin S$.

A topological space X is called *supercompact* if X has a *binary* subbase, i.e. a subbase $\mathcal{S}$ such that for each $\mathcal{S}' \subset \mathcal{S}$ the following holds: if $\cap\mathcal{S}' = \emptyset$ then $S_1 \cap S_2 = \emptyset$ for some S_1 and S_2 in $\mathcal{S}'$.
It follows from ALEXANDER's theorem that *each supercompact space is compact*. Furthermore, *each product of supercompact spaces is supercompact.*

Let $(X,\leq)$ be a lattice with universal bounds 0 and 1. If $a \in X$ and $b \in X$ then $[a,b]$ will denote the set:

$$[a,b] = \{x \in X \mid a \leq x \leq b\}.$$

Note that $[a,b] = \emptyset$ unless $a \leq b$, and $[0,1] = X$.
The *interval space* of X is the T_1-space X generated by the subbase

$$\mathcal{S} = \{[0,x] \mid x \in X,\ x \neq 1\} \cup \{[x,1] \mid x \in X,\ x \neq 0\}.$$

According to a theorem of FRINK (cf. BIRKHOFF [1]) *the interval space of* X *is compact iff* X *is complete.*
BROUWER and SCHRIJVER [2] give a characterization of supercompact spaces by means of "interval structures"; from that characterization it follows that *a compact interval space of a lattice is supercompact.*

A space X will be called *compact orderable* if it is the interval space of a complete totally ordered set X. Such a space of course is supercompact and it is also known (see e.g. KELLY [9]) that:
a compact orderable space X *is connected iff there are no neighbours in the ordering of* X, *i.e. for all* $x,y \in X$ *with* $x < y$ *there exists a* $z \in X$ *with* $x < z < y$.

b. *Graphs.*

An (undirected) *graph* G (without loops) is a pair (V,E), such that V is a (finite or infinite) set and $E \subset \{\{v,w\} \mid v,w \in V \text{ and } v \neq w\}$. The elements of V are called *vertices* and the elements of E *edges* of G. A subset V' of V is called *independent* if $\{v,w\} \notin E$ for all $v,w \in V'$; V' is called *maximal independent* if V' is maximal (under inclusion) with regard to this property.

ZORN's lemma tells us that every independent subset of V is contained in some maximal independent subset.
We write:

$$\mathcal{I}(G) = \{V' \subset V \mid V' \text{ maximal independent in } G\};$$

for each $v \in V$:

$$B_v = \{V' \in \mathcal{I}(G) \mid v \in V'\};$$

and

$$\mathcal{B}(G) = \{B_v \mid v \in V\}.$$

The *graph space* T(G) of G is the topological space with $\mathcal{I}(G)$ as underlying point set and with $\mathcal{B}(G)$ as a closed subbase.

A *path between* v *and* w ($v,w \in V$) is a collection of edges

$$\{\{v,v_1\},\ldots,\{v_k,w\}\}$$

(or the empty collection if $v = w$) with $v_1,\ldots,v_k \in V$. The *length* of a path is the number of its elements. A subset $V' \subset V$ is called *connected* if for all $v,w \in V'$ there exists a path between v and w; a *component* of G is a maximal (under inclusion) connected subset of V. We often identify a subset V' of V and the subgraph (V',E') of G, where $E' = E \cap \{\{v,w\} \mid v,w \in V'\}$, and therefore we can speak of a component of G as a subgraph of the graph.
A graph (V,E) is *bipartite* if for each $v \in V$ all paths between v and v have an even number of edges; this is the case *iff* V *is the union of two independent subsets*. For a bipartite graph (V,E) we define an equivalence relation $\sim$ on V as follows:
if $v,w \in V$ then $v \sim w$ iff there is a path between v and w of even length.

For every graph (V,E) we define a pre-order $\preccurlyeq$ (a reflexive and transitive relation) on V in the following manner:

$$v \precsim w \text{ iff for each } u \in V : \{u,v\} \in E \text{ implies } \{u,w\} \in E.$$

A graph $G = (V,E)$ is called *countable* if V is countable.
G is *comparable* if for all $v_1,v_2,v_3,v_4,v_5 \in V$ with $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_4,v_5\} \in E$ either $\{v_1,v_4\} \in E$ or $\{v_2,v_5\} \in E$.
This means that configurations as in figure 1 cannot occur in G.

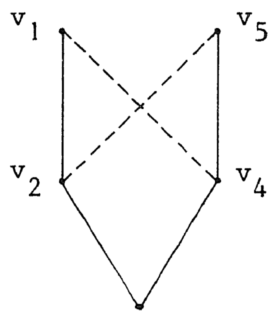

Fig.1.: *A line connecting points* x *and* y *means*: $\{x,y\} \in E$.
A dotted line between x *and* y *means*: $\{x,y\} \notin E$.

We call G *co-contiguous* if for all $\{v_2,v_3\} \in E$ there are $v_1,v_4 \in V$ with:

$$\{v_1,v_2\}, \{v_3,v_4\} \in E \quad \text{and} \quad \{v_1,v_4\} \notin E \quad \text{(see figure 2).}$$

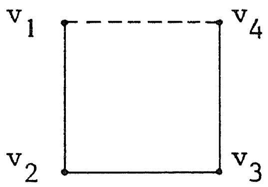

Fig.2.

In the next theorem we give a characterization of comparable graphs.

THEOREM 1.1. *Let* G *be a graph. Then* G *is comparable iff* G *is bipartite and every* $\sim$*-equivalence-class is a* $\precsim$*-chain.*

PROOF. Let $G = (V,E)$ be a comparable graph. Then G is bipartite since there are no paths between v and v of odd length ($v \in V$). For else there exists a shortest path of odd length between a vertex v and itself, say:

$$\{\{v,v_1\},\ldots, \{v_k,v\}\}.$$

If $k \geq 4$, then the comparability of G implies that:

$$\{v,v_3\} \in E \quad \text{or} \quad \{v_1,v_4\} \in E;$$

therefore an even shorter path of odd length between v and v exists after all.
If $k = 2$, then:

$$\{v,v_1\},\ \{v_1,v_2\},\ \{v_2,v\},\ \{v,v_1\} \in E$$

and then, again by the comparability of G:

$$\{v,v\} \in E \quad \text{or} \quad \{v_1,v_1\} \in E,$$

but this is not possible.
Furthermore, there are no paths of lengths 1 between v and v.
Hence G is bipartite.
As above, by the comparability of G, if there is a path of even length between v and w, then $v = w$ or there is a path of length 2 between v and w ($v,w \in V$).
Suppose now, for some $v,w \in V$, that:

$$v \sim w,\ v \not\preccurlyeq w \text{ and } w \not\preccurlyeq v.$$

Then there are x, y and z in V with:

$$\{x,v\},\ \{v,y\},\ \{y,w\},\ \{w,z\} \in E \text{ and } \{x,w\} \notin E \text{ and } \{v,z\} \notin E.$$

But this contradicts the comparability of G.

Conversely let $G = (V,E)$ be a bipartite graph, every $\sim$-class of which is an $\preccurlyeq$-chain. Take further $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_4,v_5\} \in E$. Then $v_2 \sim v_4$ and hence $v_2 \preccurlyeq v_4$ or $v_4 \preccurlyeq v_2$. If $v_2 \preccurlyeq v_4$ then $\{v_1,v_4\} \in E$ and if

$v_4 \preccurlyeq v_2$ then $\{v_2,v_5\} \in E$. Therefore, G is comparable. □

Finally let J be a set and for each $j \in J$ let $G_j = (V_j,E_j)$ be a graph. In addition, we assume that the sets $\{V_j \mid j \in J\}$ are pairwise disjoint (if necessary, the V_j have to be replaced by disjoint copies, e.g. replace V_j by $V_j \times \{j\}$).
Then the *union* $\sum_{j\in J} G_j$ of $\{G_j \mid j \in J\}$ is the graph $(\bigcup_{j\in J} V_j, \bigcup_{j\in J} E_j)$. It is clear that *each graph is the union of its components* and that *the union of bipartite, resp. comparable, resp. co-contiguous graphs is again a bipartite, resp. comparable, resp. co-contiguous graph.*

For further information concerning graphs see e.g. HARARY [8].

2. GRAPHS AND TOPOLOGICAL SPACES

In this section we study some relations between graphs and topological spaces. First we observe an obvious relation between the sum of graphs and the product of spaces.

THEOREM 2.1. *Let* J *be a set and for each* $j \in J$ *let* G_j *be a graph. Then*:

$$T(\sum_{j\in J} G_j) = \prod_{j\in J} T(G_j).$$

PROOF. Straightforward. □

THEOREM 2.2. *A space is a supercompact* T_1*-space iff it is the graph space of a graph.*

PROOF (cf. DE GROOT [6]). Let X be a supercompact T_1-space with binary subbase $\mathcal{S}$. Define a graph G with vertex set $\mathcal{S}$ and edge set E, such that: for all $S_1,S_2 \in \mathcal{S}$: $\{S_1,S_2\} \in E$ iff $S_1 \cap S_2 = \emptyset$.
Then the function $f : X \to \mathcal{I}(G)$, defined by:

$$f(x) = \{S \in \mathcal{S} \mid x \in S\}$$

is a well-defined bijection. Furthermore, for each $S \in \mathcal{S}$:

$$f[S] \in \mathcal{B}(G) \quad \text{and} \quad f^{-1}[B_S] \in \mathcal{S},$$

hence f is a homeomorphism between X and the graph space of G.

Conversely, let $G = (V,E)$ be a graph and let $X = T(G) = (\mathcal{I}(G),\mathcal{B}(G))$ be its graph space. Then X is a T_1-space, since for each $I \in \mathcal{I}(G)$ we have:

$$\{I\} = \cap\{B_v \mid v \in I\}.$$

X also is supercompact with binary subbase $\mathcal{B}(G)$, for suppose:

$$V' \subset V \text{ and } \cap\{B_v \mid v \in V'\} = \emptyset.$$

This means that there is no $I \in \mathcal{I}(G)$ such that $I \in B_v$ for each $v \in V'$, i.e. such that $V' \subset I$. But then V' is not independent and there are v_1 and v_2 in V' with $\{v_1,v_2\} \in E$; it follows that:

$$B_{v_1} \cap B_{v_2} = \emptyset. \ \square$$

<u>THEOREM 2.3</u>. *A space is a supercompact* C_{II} T_1*-space iff it is the graph space of a countable graph.*

<u>PROOF</u>. Since in a C_{II}-space every (binary) subbase contains a countable (binary) subbase this theorem is a trivial consequence of theorem 2.2. □

If DE GROOT's conjecture, that every compact metric space is supercompact, is true, then the following is also true.

<u>CONJECTURE 2.4</u>. *A Hausdorff space is compact and metrizable iff it is the graph space of a countable graph.*

"<u>PROOF</u>" (cf. DE GROOT [6]). A compact metric space is a supercompact (?) C_{II} T_1-space and hence (theorem 2.3) the graph space of a countable graph. Conversely, the graph space of a countable graph is a compact C_{II}-space. If this space is Hausdorff, it is compact and metrizable. □

<u>THEOREM 2.5</u>. *A space is the interval space of a complete lattice iff it is the graph space of a bipartite graph.*

PROOF. Let $(X,\leq)$ be a complete lattice, with universal bounds 0 and 1. In the same way as in the proof of theorem 2.2 we construct the graph with for its vertex set the binary subbase:

$$S = \{[0,x] \mid x \in X,\ x \neq 1\} \cup \{[x,1] \mid x \in X,\ x \neq 0\}$$

of the interval space of this lattice. It is immediate that this graph is bipartite.

Conversely, let $G = (V,E)$ be a bipartite graph and let $V = V_1 \cup V_2$, such that V_1 and V_2 are disjoint independent subsets of V. Define on V_1 a closure operation $V' \to \overline{V'}$ in the following manner:

$$\overline{V'} = \{x \in V_1 \mid \forall y \in V_2 : \text{if } \{y\} \cup V' \text{ is independent then } \{x,y\} \notin E\}.$$

This is indeed a closure operation since it satisfies:

(i) $V' \subset \overline{V'}$,

(ii) $\overline{V'} = \overline{\overline{V'}}$, and

(iii) if $V' \subset V''$ then $\overline{V'} \subset \overline{V''}$.

Let $\mathcal{V}$ be the collection $\{V' \subset V_1 \mid \overline{V'} = V'\}$, i.e. the collection of closed subsets of V_1; this forms a Moore family (see BIRKHOFF [1]) and therefore a complete lattice $(\mathcal{V},\leq)$, with $V' \leq V''$ iff $V' \subset V''$.
Now define a bijection $F : \mathcal{I}(G) \to \mathcal{V}$ by:

$$F(I) = I \cap V_1\ , \qquad (I \in \mathcal{I}(G)).$$

This function is well-defined, since for each $I \in \mathcal{I}(G)$ one has that:

$$\overline{I \cap V_1} = I \cap V_1\ ;$$

for suppose there is an $x \in \overline{I \cap V_1} \setminus (I \cap V_1)$. Then, since I is maximal independent, there is a $y \in I \cap V_2$ with $\{x,y\} \in E$; but $x \in \overline{I \cap V_1}$ and therefore $\{y\} \cup (I \cap V_1)$ is not independent. It follows that $I = (I \cap V_1) \cup (I \cap V_2)$ is not independent, which is a contradiction.

Also, the function F is one-to-one: if I_1 and I_2 are maximal independent and $I_1 \cap V_1 = I_2 \cap V_1$, then $I_1 \cap V_2 = I_2 \cap V_2$, and hence $I_1 = I_2$. For suppose e.g. $y \in (I_2 \cap V_2) \setminus (I_1 \cap V_2)$. Then there exists an $x \in I_1 \cap V_1$ with $\{x,y\} \in E$; but then $x \in I_2 \cap V_1$ and $y \in I_2 \cap V_2$. This contradicts the fact that $I_2 = (I_2 \cap V_1) \cup (I_2 \cap V_2)$ is independent.
Finally F is also a surjection. Indeed, if $V' \subset V_1$ and $V' = \overline{V'}$ then:

$$I := V' \cup \{y \in V_2 \mid \{x,y\} \notin E \text{ for all } x \in V'\}$$

is maximal independent and $F(I) = V'$.
Consequently, the order $\leq$ of $\mathcal{V}$ induces a complete lattice structure $(\mathcal{I}(G),\leq)$. We conclude the proof by showing that the interval topology of this lattice coincide with the topology of the graph space of G.
Let 0 and 1 be the universal bounds of $(\mathcal{I}(G),\leq)$ and take $I \in \mathcal{I}(G)$. We have to prove that $[0,I]$ and $[I,1]$ are closed in the graph space of G. But it can be seen easily that:

$$[0,I] = \{I' \in \mathcal{I}(G) \mid I' \supset I \cap V_2\} = \bigcap\{B_v \mid v \in I \cap V_2\},$$

and

$$[I,1] = \{I' \in \mathcal{I}(G) \mid I' \supset I \cap V_1\} = \bigcap\{B_v \mid v \in I \cap V_1\};$$

therefore $[0,I]$ and $[I,1]$ are closed in the graph space.
Conversely, take $v \in V$. If $v \in V_1$ then:

$$B_v = \{I \in \mathcal{I}(G) \mid v \in I\} = \left[\overline{\{v\}},1\right],$$

and if $v \in V_2$ then:

$$B_v = \{I \in \mathcal{I}(G) \mid v \in I\} = [0,J]$$

where:

$$J = \{x \in V_1 \mid \{x,v\} \notin E\} \cup \{y \in V_2 \mid \forall x \in V_1 : \{x,y\} \in E \rightarrow \{x,v\} \in E\}.$$

So the two topologies coincide. □

Theorem 2.5 has the following consequence.

COROLLARY 2.6. *Let* X *be a* T_1*-space. Then the following assertions are equivalent:*

(i) X *is the interval space of a complete lattice;*

(ii) X *has a binary subbase* S *such that whenever* $S_1,\dots,S_k \in S$ *and* $S_1 \cap S_2 = S_2 \cap S_3 = \dots = S_{k-1} \cap S_k = S_k \cap S_1 = \emptyset$ *it follows that* k *is even ;*

(iii) X *has a binary subbase* S *and there are two points* x *and* y *such that each* $S \in S$ *contains either* x *or* y.

PROOF. Since a graph $G = (V,E)$ is bipartite if and only if for each $v \in V$ the paths between v and v are of even length, and also if and only if V is the union of two independent subsets of V, the corollary follows easily from theorem 2.5. □

In theorem 2.5 we proved that the class of graph spaces of bipartite graphs coincides with the class of interval spaces of complete lattices. In the next theorems we give similar relations between graph spaces of bipartite graphs of some special kind and interval spaces of complete lattices of some special kind.

THEOREM 2.7. *A space is a compact orderable space iff it is the graph space of a connected comparable graph.*

PROOF. Let $(X,\leq)$ be a complete totally ordered set with universal bounds 0 and 1. A subbase for its interval topology is:

$$S = \{[0,x] \mid x \in X, x \neq 1\} \cup \{[x,1] \mid x \in X, x \neq 0\}.$$

The graph $G = (S,E)$ generated by this subbase clearly is bipartite, connected and comparable (there are only two ~-equivalence classes, each of which is a $\preccurlyeq$-chain).

Conversely, let $G = (V,E)$ be a connected comparable graph. Let $V = V_1 \cup V_2$, with V_1 and V_2 disjoint and independent. Since G is connected, G has only two ~-classes, namely V_1 and V_2 (the case that there is only one class, i.e. $|V| = 1$, is trivial). We now proceed as in the proof of theorem 2.5. Consider again:

$$\mathcal{V} = \{V' \subset V_1 \mid \overline{V'} = V'\}$$

and let $\leq$ be the order induced by the inclusion relation on $\mathcal{V}$. Then $(\mathcal{V},\leq)$ is again a complete lattice; we will show that in this case it is also a chain. Suppose, to the contrary, that it is not a chain. Then there are $V', V'' \subset V_1$ with:

$$\overline{V'} = V' \quad \text{and} \quad \overline{V''} = V'',$$

and vertices v_1 and v_2 such that:

$$v_1 \in V' \setminus V'' \quad \text{and} \quad v_2 \in V'' \setminus V'.$$

This means that there are w_1 and w_2 in V_2 such that:

$$\{v_1,w_1\},\ \{v_2,w_2\} \in E,$$

and:

$$\{v,w_2\} \notin E \text{ for all } v \in V',$$
$$\{v,w_1\} \notin E \text{ for all } v \in V''.$$

But then, in particular:

$$\{v_1,w_2\} \notin E \quad \text{and} \quad \{v_2,w_1\} \notin E.$$

Since G is comparable, we know that $v_1 \preccurlyeq v_2$ or $v_2 \preccurlyeq v_1$. However, this is in contradiction with the existence of w_1 and w_2 as described above. □

<u>THEOREM 2.8</u>. *A space is a product of compact orderable spaces iff it is the graph space of a comparable graph.*

<u>PROOF</u> (cf. DE GROOT & SCHNARE [8]). Combine theorems 2.1. and 2.7. □

<u>THEOREM 2.9</u>. *A space is a connected compact orderable space iff it is the graph space of a connected co-contiguous comparable graph.*

<u>PROOF</u>. Let $(X,\leq)$ be a complete chain without neighbours and with universal bounds 0 and 1. Let:

$$S = \{[0,x] \mid x \in X,\ x \neq 1\} \cup \{[x,1] \mid x \in X,\ x \neq 0\}$$

be the subbase for the order topology of X considered before in the proof of theorem 2.7. As in that proof we construct the connected and comparable graph $G = (S,E)$. We will show that this graph is also co-contiguous. Take $\{[0,x], [y,1]\} \in E$, i.e. $[0,x] \cap [y,1] = \emptyset$ and therefore $x < y$. Since X has no neighbours there is some z with $x < z < y$ and hence:

$$\{[0,x], [z,1]\} \in E,$$

$$\{[0,z], [y,1]\} \in E,$$

and:

$$\{[0,z], [z,1]\} \notin E.$$

So G is co-contiguous.

Conversely, let $G = (V,E)$ be a connected co-contiguous and comparable graph and let V_1 and V_2 be its two ~-equivalence-classes. It follows from theorem 2.7. that the graph space of G is compact orderable. Define, as in the proof of theorem 2.5.:

$$\mathcal{V} = \{V' \subset V_1 \mid \overline{V'} = V'\}$$

and let $\leq$ be the order induced by the inclusion relation on $\mathcal{V}$. To prove connectedness of the graph space of G it is sufficient to show that $\mathcal{V}$ has no neighbours. Suppose $\mathcal{V}$ has neighbours V' and V" and V' < V". Take

$v_1 \in V'' \setminus V'$ and let $w_1 \in V_2$ such that $\{v_1,w_1\} \in E$ and $\{v, w_1\} \notin E$ for all $v \in V'$. Since G is co-contiguous there are $v_2 \in V_1$ and $w_2 \in V_2$ with:

$$\{v_2,w_1\},\ \{v_1,w_2\} \in E \quad\text{and}\quad \{v_2,w_2\} \notin E.$$

From this it follows that:

$$v_2 \lesssim v_1 \quad\text{and}\quad v_2 \in V'' \setminus V'.$$

Also it can be seen easily that:

$$v_1 \notin \overline{V' \cup \{v_2\}} \quad\text{and}\quad V' < \overline{V' \cup \{v_2\}} < V''.$$

This contradicts the assumption that V' and V" are neighbours. □

<u>THEOREM 2.10</u>. *A space is a product of connected compact orderable spaces iff it is the graph space of a co-contiguous comparable graph.*

<u>PROOF</u>. Combine theorems 2.1. and 2.9. □

<u>THEOREM 2.11</u>. *A space is homeomorphic to a compact subset of the real line iff it is the graph space of a connected comparable countable graph.*

<u>PROOF</u>. Let X be a compact subset of the real line. We may suppose:

$$X \subset I,$$

where I is the unit segment. Let:

$$\mathcal{S} = \{[0,x] \cap X \mid x \in X\} \cup \{[x,1] \cap X \mid x \in X\} \setminus \{X\}$$

be a subbase for X. Since X is a C_{II}-space, $\mathcal{S}$ contains a countable subbase $\mathcal{S}'$. Then the graph $(\mathcal{S}',E)$, as defined in the proof of theorem 2.2, is connected, comparable and countable.

Conversely, the graph space of a connected comparable countable graph is, by theorems 2.3 and 2.7., compact orderable and C_{II} and hence homeomorphic to a compact subspace of the real line. □

THEOREM 2.12. *A space is homeomorphic to a product of compact subspaces of the real line iff it is the graph space of a comparable graph, each component of which is countable.*

PROOF. Combine theorems 2.1. and 2.12. □

THEOREM 2.13. *A space is homeomorphic to the unit segment* I *iff it is the graph space of a connected comparable co-contiguous countable graph with more than one vertex.*

PROOF. Let D be a countable dense subset of I and put:

$$S = \{[0,x] \mid x \in D, x \neq 1\} \cup \{[x,1] \mid x \in D, x \neq 0\}.$$

Construct the graph (S,E) as in theorem 2.2.
From the proofs of the preceding theorems it is clear that this graph is connected, comparable, co-contiguous and countable and that it has more than one vertex.

Conversely, let X be the graph space of a connected comparable co-contiguous countable graph with more than one vertex. Then it follows from theorem 2.11. that X is homeomorphic to a compact subset of I and from theorem 2.9. that X is connected. Since the graph has more than one vertex and is connected, X has more than one point and so X is homeomorphic to I. □

THEOREM 2.14. *A space is homeomorphic to a product of intervals* I *iff it is the graph space of a comparable co-contiguous graph each component of which is countable.*

PROOF (cf. DE GROOT [6] and BRUIJNING [3]). This is a consequence of theorems 2.1. and 2.13. We remark that the one-point-space can be considered as an empty product. □

Finally we give a characterization of graphs G, the graph space of which is a Hausdorff space.
First we prove:

<u>THEOREM 2.15</u>. *Let* X *be a supercompact* T_1*-space with binary subbase* $\mathcal{S}$. *Then* X *is a Hausdorff space iff for all* $S_1, S_2 \in \mathcal{S}$ *with:*

$$S_1 \cap S_2 = \emptyset$$

there are

$$R_1, \ldots, R_k,\ Q_1, \ldots, Q_\ell \in \mathcal{S} \qquad (k,\ \ell \geq 0)$$

with:

$$(R_1 \cup \ldots \cup R_k) \cap S_1 = (Q_1 \cup \ldots \cup Q_\ell) \cap S_2 = \emptyset$$

and:

$$R_1 \cup \ldots \cup R_k \cup Q_1 \cup \ldots \cup Q_\ell = X.$$

<u>PROOF</u>. It is well-known that a compact T_1-space is a Hausdorff space iff it is a normal space.
Let X be a normal space and take $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \emptyset$. Then there are closed C and D with:

$$C \cap S_1 = D \cap S_2 = \emptyset \quad \text{and} \quad C \cup D = X.$$

Since X is compact and C and D are intersections of finite unions of sets in $\mathcal{S}$, we can take as C and D finite intersections of finite unions of sets in $\mathcal{S}$, or, what is the same, finite unions of finite intersections of sets in $\mathcal{S}$.
Since $C \cap S_1 = \emptyset$, each of the finite intersections composing C has an empty intersection with S_1. Now $\mathcal{S}$ is binary and therefore we can replace these finite intersections by single sets of $\mathcal{S}$. Hence we may suppose that C is a finite union of sets in $\mathcal{S}$, i.e.:

$$C = R_1 \cup \ldots \cup R_k \qquad (k \geq 0,\ R_1, \ldots, R_k \in \mathcal{S}).$$

Similarly we can take:

$$D = Q_1 \cup \ldots \cup Q_\ell \qquad (\ell \geq 0,\ Q_1,\ldots,Q_\ell \in S).$$

This proves one side of the theorem.

Conversely, suppose S satisfies the conditions stated above. We now prove that X is normal. Let E and F be closed subsets of X with $E \cap F = \emptyset$. Again, since X is compact, we may suppose that E and F are finite intersections of finite unions of sets in S, or finite unions of finite intersections. So every finite intersection composing E is disjoint from every finite intersection composing F. Hence, since S is binary, there are disjoint sets of S in every such pair, and therefore, by the condition on S, there are disjoint open neighbourhoods for every pair. It can now be seen easily that E and F have also disjoint open neighbourhoods. Thus X is normal. □

A consequence of this is:

<u>COROLLARY 2.16</u>. *Let* $G = (V,E)$ *be a graph. Then the graph space* X *of* G *is a Hausdorff space iff for each* $\{v,w\} \in E$ *there are* $v_1,\ldots,v_k,\ w_1,\ldots,w_\ell \in V$ $(k,\ell \geq 0)$ *such that:*

$$\{v,v_1\},\ldots,\{v,v_k\},\ \{w,w_1\},\ldots,\{w,w_\ell\} \in E,$$

and if:

$$v_1',\ldots,v_k',\ w_1',\ldots,w_\ell' \in V$$

with: $$\{v_1,v_1'\},\ldots,\{v_k,v_k'\},\ \{w_1,w_1'\},\ldots,\{w_\ell,w_\ell'\} \in E,$$

then

$$\{v_1',\ldots,v_k',\ w_1',\ldots,w_\ell'\}$$

is not independent.

PROOF. This follows from theorem 2.15, applied to the space $I(G)$ and the subbase $\mathcal{B}(G)$. For the subbase elements S_1, S_2, R_i, Q_j mentioned in theorem 2.15. we determine vertices v, w, v_i, w_j such that:

$$B_v = S_1,\ B_w = S_2,\ B_{v_1} = R_1,\dots,B_{v_k} = R_k,\ B_{w_1} = Q_1,\dots,B_{w_\ell} = Q_\ell.$$

In addition, we observe that:

$$B_{v_1} \cup\dots\cup B_{v_k} \cup B_{w_1} \cup\dots\cup B_{w_\ell} = X$$

iff there is no maximal independent set V' in V with:

$$V' \cap \{v_1,\dots,v_k, w_1,\dots,w_\ell\} = \emptyset,$$

i.e. iff whenever $V' \subset V$ and V' is independent, there is a $u \in \{v_1,\dots,v_k, w_1,\dots,w_\ell\}$ such that $V' \cup \{u\}$ is independent. □

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